OLIVIER'S THEOREM DEBUNKED …

The mathematical activity is about developing a **habit (attitude) of mind:** we must be careful not to *assume* that something is true just because it *looks* like it*!*

"*Geometry is the science of correct reasoning on incorrect figures".* George Pólya, How to Solve It, 1945, p. 208.

"*We must doubt everything, we cannot trust our senses*". René Descartes, 1637.

Fact is that despite our Figure 1, the point O never lies *inside* the triangle!
Do you agree? How would you convince yourself, or someone else, that this assertion is true?

But that is not the error! Our "proof" does cover the case where O lies outside the triangle*!*

The fundamental error in our proof is in the assumption that P and Q *both* are *on* the triangle or *both* are *outside* the triangle. Although it *looks* like it in our sketches, is not true!

Dynamic Geometry software such as *Geometer's Sketchpad* and *Geogebra* are useful tools for us to check such conjectures. You will make the necessary construction (Figure 4)!

So, *everything in our proof is correct,* except for the penultimate step:

(3) + (4): AP + PB = AQ + QC
or
(3) – (4): AP – PB = AQ – QC

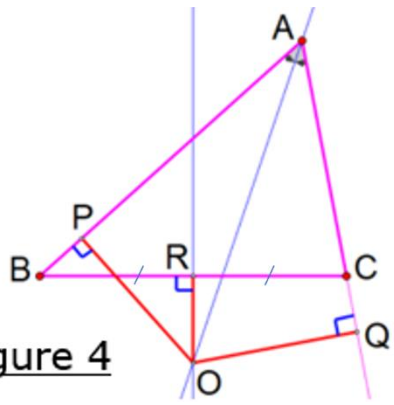


Figure 4

Fact is: one of P or Q is *on* the triangle and the other is *outside*, as
Then (3) + (4): AP + PB (= AB) = AQ + QC ($\neq$ AC)
or (3) – (4): AP – PB ($\neq$ AB) = AQ – QC (= AC)
So the final conclusion that AB = AC *cannot* be made!!

But if our conclusion that the construction is incorrect is based on <u>how it looks</u> in a more accurate sketch or even in a construction, we make exactly the same thinking error as in the original!
To be <u>sure</u>, we have to reason <u>deductively</u>, i.e. generally, without using any specific measurements!
We cannot construct or measure accurately, so we cannot perceive (see) accurately!

We cannot trust our senses!

We will illustrate with more examples …

To <u>*prove*</u> *that the sketches used in the proof are wrong, we must* <u>*prove*</u> *that P and Q lie on opposite sides of BC.*

This proof requires knowledge of cyclic quadrilaterals which is in the FET syllabus, so this content is not meant for learners, but must be seen as teacher professional development and enrichment.

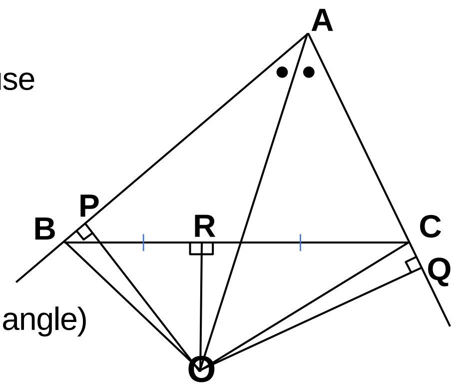
There is a known theorem, the (Wallace-)Simson theorem, stating:

The feet of the perpendiculars from a point to the sides of a triangle are colinear if and only if the point lies on the circumcircle.

All that we therefore have to do, is to prove that our constructed point O (i.e. is the intersection of the bisector of ∠A and the perpendicular bisector of BC) lies on the circumcircle of ΔABC. Then it follows from the Simson theorem that PRQ is a straight line, from which it follows that P and Q lie on *opposite* sides of BC, so all three figures we used in the proof are wrong, and therefore the reasoning is wrong!

<u>Prove that O lies on the circumcircle ABC</u>

My proof-plan is to prove ACOB is a cyclic quadrilateral because exterior angle OCQ = opposite interior angle OBP:



Prove exactly like in the original proof
that ΔCQO ≡ ΔBPO (90°, hyp, s)

⇒ ∠OCQ = ∠OBP
⇒ ACOB is a cyclic quadrilateral (Exterior angle = opp interior angle)
⇒ O lies on circumcircle ABC

If we accept the Simson theorem, we can now deduce that P, R and Q are colinear (therefore the construction in the equilateral triangle "proof" is wrong!).

But if you do not believe Simson, let's prove it!

<u>Prove that PRQ is a straight line</u>

We will now prove that *if* O lies on the circumcircle of ΔABC (proved above), *then* P, R and Q lie on a straight line. Then we can conclude that P and Q lie on opposite sides of BRC, and then we have proven that our Figure 1, Figure 2 and Figure 3 were all wrong, and therefore our reasoning bases on the sketches were all wrong!

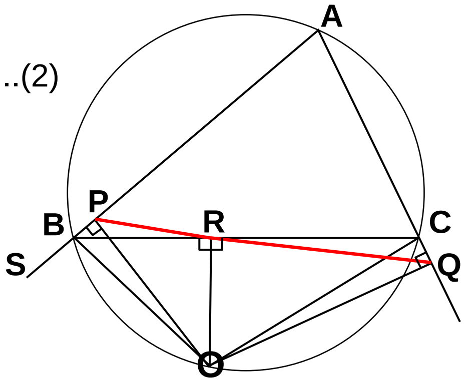
We can prove that PRQ is a straight line by proving that ∠PRQ = ∠PRO + ∠ORQ =180°

BORP IS a cyclic quadrilateral (∠BPO = ∠BRO = 90°, construction; angles on chord BO)

⇒ ∠PRO = ∠OBS … exterior angle of cyclic quad BORP ………...….(1)

CQOR is a cyclic quadrilateral (∠OQC = ∠ORC = 90°, construction; exterior angle equals opp int angle)

⇒ ∠ORQ = ∠OCQ … angles on chord OQ of cyclic quad CQOR
= ∠OBA … exterior angle of cyclic quad ACOB …...(2)



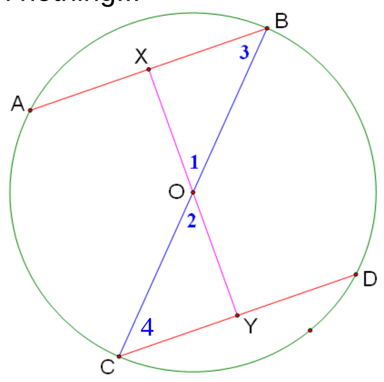
From (1) and (2):
∠PRO + ∠ORQ = ∠OBS + ∠OBA = 180° … straight line SBA

⇒ PRQ is a straight line

IMPLICATIONS

We should be careful in riders not to draw *special cases* and then unthinkingly make assumptions!

What is wrong with such assumptions?
When using the Modus Ponens proof structure, we say P $\Rightarrow$ Q, P is true, so Q is true. But if it turns out that P is not true (we assumed something that is false!), we have proven nothing!!!



Here is a simple example …

CONJECTURE: In any circle, equal chords are equidistant from the centre.

Construction: Draw chords AB = CD, and draw OX $\perp$ AB and OY $\perp$ CD

Required to prove: OX = OY

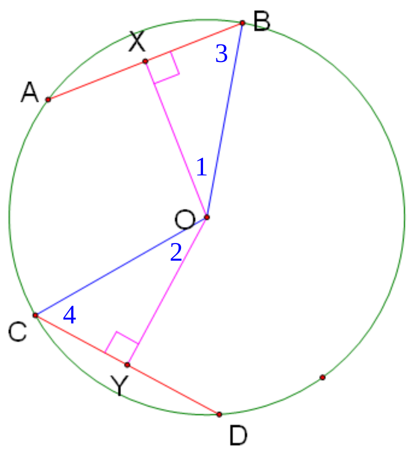
In each case below, say if the proof is valid or not:

1. ΔOXB $\equiv$ ΔOYC ($\angle$, $\angle$, s .. $\angle$1 = $\angle$2 vertical angles, $\angle$3 = $\angle$4 alternate angl
   $\Rightarrow$ OX = OY
2. ΔOXB $\equiv$ Δ OYC ($\angle$, $\angle$, s .. $\angle$1 = $\angle$2 vertical angles, $\angle$X = $\angle$Y = 90° construction, OB = OC)
   $\Rightarrow$ OX = OY
3. ΔOXB $\equiv$ Δ OYC (90°, hyp, s ... $\angle$X = $\angle$Y = 90° construction, OB = OC radii, ½ AB = ½ CD)
   $\Rightarrow$ OX = OY
4. ΔOXB $\equiv$ Δ OYC (s, $\angle$, s .. OB = OC radii, $\angle$3 = $\angle$4 alternate angles, ½ AB = ½ CD)
   $\Rightarrow$ OX = OY
5. XB = CY (½ AB = ½ CD) and XB || CY $\Rightarrow$ XBYC is a parallelogram (one pair opposite sides equal and parallel)
   $\Rightarrow$ OX = OY (diagonals bisect each other)

None of the above proof attempts are valid!

Most proofs assume something special in the sketch, i.e. that the chords are parallel (so $\angle$3 = $\angle$4) or that the lines were straight (so $\angle$1 = $\angle$2) or that AX = XB and CY = *YD because it looks so!*

This is a much better, general sketch, leading to a correct proof!



$\angle$X = $\angle$Y = 90° ……. construction
OB = OC ....radii
XB = CY ...... ½ AB = ½ CD (theorem: line from centre
perpendicular to chord bisects the chord)
$\Rightarrow$ ΔOXB $\equiv$ ΔOYC (90°, hyp, s)
$\Rightarrow$ OX = OY

Note again, that even though it turns out to be *true* that $\angle$3 = $\angle$4 and $\angle$1 = $\angle$2, and AX = XB, it is not *proven* by the arguments in any of the above proof attempts, that is why those proofs are invalid!

Do you have examples from your own experience?

Drawing special cases is one source of making invalid assumptions.
But we make such assumptions all the time!
We must learn the habit of mind to doubt and check the validity of each step!!